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Representation of the Einstein–Proca field by an A_4^*

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Abstract. The Einstein–Proca field is 'geometrised' in the sense that the joint field equations are generated by a variational principle, the Lagrangian of which is solely a certain function of curvature quantities and the metric tensor g_{ij} of an A_4^*, i.e. of a linearly connected space A_4 with symmetric connection on which an additional symmetric tensor field g_{ij} is defined.

1. Introduction

Formally the general theory of relativity may be regarded as achieving the representation of gravitational fields by a 4-dimensional Riemann space V_4 subject to the condition that the action, i.e. the integral of the scalar curvature density $\mathfrak{R}$, taken over an arbitrary region, be stationary with respect to arbitrary infinitesimal variations of the components g_{ij} of the metric tensor (of signature -2) which vanish on the boundary. In this sense the Einstein–Maxwell field may be represented by a 5-dimensional Riemann space V_5, again subject to the condition of the stationarity of the action as previously defined, but with an additional generic condition imposed upon the metric tensor. This theory—the Klein–Kaluza theory (Pauli 1958)—then replaces the original 4-dimensional theory in which the electromagnetic field formally appears as a source; that is to say, in which its presence is allowed for by the *ad hoc* addition to $\mathfrak{R}$ of an extrinsic 'matter Lagrangian' representing this source. An analogous situation prevails with respect to massless scalar fields. In either case one sometimes speaks loosely of the 'geometrisation' of the theory in question; and it is worthy of note that each time the source field is *massless*.

The variations contemplated above were those of the metric tensor: one has 'g-variations'. Occasionally one substitutes for these so-called 'P-variations': the Christoffel symbols are simply replaced by the components $\Gamma^m{}_{kl}$ of a symmetric linear connection, and the g_{ij} and $\Gamma^m{}_{kl}$ are then subjected to arbitrary infinitesimal variations, independently of each other, the action now being required to be stationary with respect to these. Now, leaving aside the fact that for an arbitrary chosen Lagrangian P-variations may lead to various difficulties (cf Buchdahl 1960), P-variation and g-variation are far from being mutually equivalent, very special Lagrangians apart. I have discussed this inequivalence in some detail in the preceding paper (Buchdahl 1979). It suffices to remark here that, whereas in the context of g-variations one deals with a Riemann space from the outset, in the context of P-variations one contemplates in the first place a linearly connected space with symmetric connection on which an independent covariant tensor field g_{ij} of signature -2 (a 'metric tensor') is defined. Such a space will be referred to as an A_4^*. There is then no reason why the P-equations should necessarily imply that the A_4^* is metric with respect to g_{ij} or, indeed, metric with respect to some other symmetric tensor $\bar{g}_{ij}$. In short, a specific theory is defined by the Lagrangian *together with* the prescription as to whether one is to carry out g-variations or P-variations.

In this paper I contemplate P-variations alone. This brings with it a certain greater degree of generality than would otherwise be the case, in as far as the skew-symmetric part $R_{[ij]}$ of the Ricci tensor of the A_4^* is now available—for instance in the context of the construction of Lagrangians—whereas $R_{[ij]}$ necessarily vanishes in a V_4. In particular I investigate the consequences of the choice

$$L = R + \alpha R_{[kl]} R^{kl} \qquad (\alpha = \text{constant}). \tag{1.1}$$

Remarkably, it turns out that it leads precisely to the Einstein–Proca equations, where it should also be borne in mind that the Proca equations govern a *massive* vector field.

2. The field equations

Variation of the Lagrangian (1.1) with respect to the g_{ij} alone leads, virtually by inspection, to the first field equation

$$(\tfrac{1}{2}g_{ij}R - R_{(ij)}) + 2\alpha(\tfrac{1}{4}g_{ij}f_{mn}f^{mn} - f_{im}f_j{}^m) = 0, \tag{2.1}$$

where

$$f_{ij} := R_{[ij]} \equiv \Gamma_{[i,j]} \qquad (\Gamma_i := \Gamma^m{}_{mi}). \tag{2.2}$$

Indices are raised and lowered by means of g^{ij} and g_{ij} respectively as usual.

The second field equation results from the variation of the $\Gamma^k{}_{ij}$ alone. Bearing in mind that

$$\delta R^l{}_{ijk} = -2\delta\Gamma^l{}_{i[j;k]},$$

where subscripts following a semicolon denote covariant differentiation with respect to $\Gamma^r{}_{pq}$, it follows directly that

$$\mathfrak{g}^{ij}{}_{;k} - \delta^{(i}{}_k(\mathfrak{g}^{j)l}{}_{;l} + 2\alpha\mathfrak{f}^{j)l}{}_{;l}) = 0. \tag{2.3}$$

(Substitution of fraktur for italic kernel symbols implies multiplication by $w := (-g)^{1/2}$.) Equations (2.1) and (2.3) are the required 'P-equations'.

3. The connection $\Gamma^l{}_{mn}$

Equation (2.3) is now to be solved generically for the components of connection. Contraction of (2.3) shows that

$$\mathfrak{g}^{ij}{}_{;j} = -\tfrac{10}{3}\alpha\mathfrak{f}^{ij}{}_{;j} \tag{3.1}$$

and by means of this (2.3) becomes

$$\mathfrak{g}^{ij}{}_{;k} = \tfrac{2}{5}\delta^{(i}{}_k\mathfrak{g}^{j)l}{}_{;l}. \tag{3.2}$$

Written out in full this reads, with $l_k := (\ln w)_{,k}$,

$$g^{ij}{}_{,k} + \Gamma^i{}_{km} g^{mj} + \Gamma^j{}_{km} g^{mi} + g^{ij}(l_k - \Gamma_k) = \tfrac{2}{5}\delta^{(i}{}_k (g^{j)m}{}_{,m} + \Gamma^{j)}{}_{mn} g^{mn} + g^{j)m} l_m). \tag{3.3}$$

By transvection with g_{ij} one finds that

$$g^{jm}{}_{,m} + \Gamma^j{}_{mn} g^{mn} = g^{jm}(4l_m - 5\Gamma_m).$$

The expression on the left may be eliminated from the right-hand member of (3.3). The resulting equation may be written

$$\Gamma^m{}_{ik} g_{mj} + \Gamma^m{}_{jk} g_{mi} = g_{ij,k} + g_{ki} m_j + g_{kj} m_i - g_{ik} m_j,$$

with $m_j := (l_j - \Gamma_j)$. From this one infers that

$$\Gamma^k{}_{ij} = \mathring{\Gamma}^k{}_{ij} - \tfrac{1}{2}(\delta_i{}^k m_j + \delta_j{}^k m_i - 3 g_{ij} m^k),$$

where $\mathring{\Gamma}^k{}_{ij}$ are the Christoffel symbols which belong to g_{ij}. On taking the trace one obtains an identity, so that m_j remains undetermined. In other words, (2.3) implies that

$$\Gamma^k{}_{ij} = \mathring{\Gamma}^k{}_{ij} + \delta_i{}^k u_j + \delta_j{}^k u_i - 3 g_{ij} u^k, \tag{3.4}$$

where u_i is a vector which is arbitrary as far as equation (3.2) is concerned. By contraction

$$\Gamma_i = \mathring{\Gamma}_i + 2u_i, \tag{3.5}$$

so that

$$f_{ij} = 2u_{[i,j]}. \tag{3.6}$$

Also, if subscripts following a colon denote covariant differentiation with respect to $\mathring{\Gamma}^k{}_{ij}$,

$$\mathfrak{g}^{ij}{}_{;j} = \mathfrak{g}^{ij}{}_{:j} + (\Gamma^i{}_{mn} - \mathring{\Gamma}^i{}_{mn})\mathfrak{g}^{mn}.$$

Since $\mathfrak{g}^{ij}{}_{:j}$ vanishes of course, it follows on using (3.4) that

$$\mathfrak{g}^{ij}{}_{;j} = -10\mathfrak{u}^i.$$

Then (3.1) shows that u_i must satisfy the equation

$$\mathfrak{f}^{ij}{}_{;j} = 3\alpha^{-1}\mathfrak{u}^i, \tag{3.7}$$

where (3.6) is to be recalled. It follows from this incidentally that $\mathfrak{u}^i{}_{;i} = 0$ or, what comes to the same thing,

$$u^i{}_{:i} = 0. \tag{3.8}$$

In summary, $\Gamma^k{}_{ij}$ is given by (3.4), where u_i is a vector which satisfies (3.7). Note that the latter does not involve $\Gamma^k{}_{ij}$.

4. Emergence of the Einstein–Proca equations

No use has been made as yet of equation (2.1). Although transvection of this with g^{ij} shows that $R = 0$, it is convenient to carry R along. The Ricci tensor R_{ij} may be written as the sum of terms, the first of which is the Ricci tensor $\mathring{R}_{ij}$ of the V_4 whose metric tensor is g_{ij}. $\mathring{R}$ stands for the scalar curvature of this V_4. Then one finds by elementary means that

$$R_{ij} = \mathring{R}_{ij} + 2f_{ij} + 6u_i u_j, \qquad R = \mathring{R} + 6u_m u^m. \tag{4.1}$$

Inserting these in (2.1), there comes

$$\tfrac{1}{2}g_{ij}\mathring{R}-\mathring{R}_{ij}=-2\alpha(\tfrac{1}{4}g_{ij}f_{mn}f^{mn}-f_{im}f_j{}^m)+(6u_iu_j-3g_{ij}u_lu^l). \tag{4.2}$$

Put

$$u_i=:\gamma U_i, \qquad F_{ij}:=2U_{[i,j]}, \tag{4.3}$$

where γ is a constant. At the same time, express α as a matter of convenience in terms of another constant m according to

$$\alpha=-3/m^2, \tag{4.4}$$

and choose γ to have the value given by

$$\gamma=(4\pi/3)^{1/2}m. \tag{4.5}$$

Then (4.2) and (3.7) become

$$\tfrac{1}{2}g_{ij}\mathring{R}-\mathring{R}_{ij}=8\pi[(\tfrac{1}{4}g_{ij}F_{mn}F^{mn}-F_{im}F_j{}^m)+m^2(U_iU_j-\tfrac{1}{2}g_{ij}U_mU^m)] \tag{4.6}$$

and

$$\mathfrak{F}^{ij}{}_{,j}+m^2\mathfrak{U}^i=0 \tag{4.7}$$

respectively. By inspection, (4.3), (4.7) and (4.6) are exactly the equations of the Einstein–Proca field. If T_{ij} stands for the right-hand member of (4.6), $T^{ij}{}_{;j}$ must vanish. A straightforward calculation shows that it in fact does so whenever (4.7) is satisfied.

5. Concluding remark

The Proca field has been 'geometrised' along with the gravitational field in terms of an A_4^* in the sense that the equations governing the joint fields take the form $\delta\int\mathfrak{L}\,\mathrm{d}^{(4)}x=0$ for P-variations, where $\mathfrak{L}$ is made up of the metric tensor and curvature quantities of the A_4^* alone. There does not appear to exist any sensible limiting process which leads to the Einstein–Maxwell field.

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